

## Practice Problems on Integration by Parts (with Solutions)

This problem set is generated by Di. All of the problems came from the past exams of Math 222 (2011-2016). Many exam problems come with a special twist. I pick the representative ones out. For some of you who want more practice, it's a good pool of problems. The solutions are not proven to be typo-free. If there's any errors, please kindly let me know. :)

1.  $\int \arctan(3z) dz$
2.  $\int (\ln x)^2 dx$
3.  $\int x^2 \cos \frac{x}{2} dx$
4.  $\int \ln(x^2 + 1) dx$
5.  $\int \cos x \ln(\sin x) dx$
6.  $\int \sin(\sqrt{x}) dx$
7.  $\int x^3 e^{x^2} dx$      *This problem from 2016 is very very hard. Hint: try u-sub first.*
8.  $I_n = \int x^n \sin(2x) dx$
9.  $I_n = \int x^n e^{2x} dx$
10.  $I_n = \int (1+x)^n \sin x dx$
11.  $I_n = \int x^{-n} \sin x dx$

Solutions are provided in the next page.

$$1. \int \arctan(3z) dz = z \arctan 3z - \frac{1}{6} \ln |1 + 9z^2| + C$$

**Work:**

$$\begin{aligned} \int \arctan(3z) dz &= z \arctan(3z) - \int \frac{3z}{1+9z^2} dz && \left| \begin{array}{l} F = \arctan(3z), \quad G' = 1 \\ F' = \frac{3}{1+9z^2}, \quad G = z \end{array} \right. \\ &= z \arctan(3z) - \frac{1}{6} \int \frac{1}{u} du && \left| \begin{array}{l} u = 1 + 9z^2 \\ du = 18z dz \Rightarrow \frac{1}{6} du = 3z dz \end{array} \right. \\ &= z \arctan(3z) - \frac{1}{6} \ln |u| + C \\ &= z \arctan(3z) - \frac{1}{6} \ln |1 + 9z^2| + C \end{aligned}$$

$$2. \int (\ln x)^2 dx = x (\ln^2 x - 2 \ln x + 2) + C$$

**Work:**

$$\begin{aligned} \int (\ln x)^2 dx &= x (\ln x)^2 - 2 \int \ln x dx && \left| \begin{array}{l} F = (\ln x)^2 \quad G' = 1 \\ F' = 2 \ln x \cdot \frac{1}{x} \quad G = x \end{array} \right. \\ &= x (\ln x)^2 - 2 \int \ln x dx && \left| \begin{array}{l} F = \ln x \quad G' = 1 \\ F' = \frac{1}{x} \quad G = x \end{array} \right. \\ &= x (\ln x)^2 - 2 \left( x \ln x - \int 1 dx \right) \\ &= x (\ln x)^2 - 2(x \ln x - x) + C \end{aligned}$$

$$3. \int x^2 \cos \frac{x}{2} dx = 2x^2 \sin \frac{x}{2} + 8x \cos \frac{x}{2} - 16 \sin \frac{x}{2} + C$$

**Work:**

$$\begin{aligned} \int x^2 \cos \frac{x}{2} dx &= 2x^2 \sin \frac{x}{2} - 4 \int x \sin \frac{x}{2} dx && \left| \begin{array}{l} F = x^2 \quad G' = \cos \frac{x}{2} \\ F' = 2x \quad G = 2 \sin \frac{x}{2} \end{array} \right. \\ &= 2x^2 \sin \frac{x}{2} - 4 \int x \sin \frac{x}{2} dx && \left| \begin{array}{l} F = x \quad G' = \sin \frac{x}{2} \\ F' = 1 \quad G = -2 \cos \frac{x}{2} \end{array} \right. \\ &= 2x^2 \sin \frac{x}{2} - 4 \left( -2x \cos \frac{x}{2} + 2 \int \cos \frac{x}{2} dx \right) \\ &= 2x^2 \sin \frac{x}{2} - 4 \left( -2x \cos \frac{x}{2} + 4 \sin \frac{x}{2} \right) + C \end{aligned}$$

$$4. \int \ln(x^2 + 1) dx = x \ln(x^2 + 1) + 2 \arctan x - 2x + C$$

**Work:**

$$\begin{aligned} \int \ln(x^2 + 1) dx &= x \ln(x^2 + 1) - 2 \int \frac{x^2}{x^2 + 1} dx && \left| \begin{array}{l} F = \ln(x^2 + 1) \quad G' = 1 \\ F' = \frac{2x}{x^2 + 1} \quad G = x \end{array} \right. \\ &= x \ln(x^2 + 1) - 2 \int \frac{x^2 + 1 - 1}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - 2 \int 1 - \frac{1}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - 2(x - \arctan x) + C \end{aligned}$$

$$5. \int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \sin x + C$$

**Work:** **Approach 1:** IBP directly

$$\begin{aligned} \int \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) - \int \cos x dx \\ &= \sin x \ln(\sin x) - \sin x + C \end{aligned} \quad \left| \begin{array}{l} F = \ln(\sin x) \quad G' = \cos x \\ F' = \frac{\cos x}{\sin x} \quad G = \sin x \end{array} \right.$$

**Approach 2:** u-sub + IBP

$$\begin{aligned} \int \cos x \ln(\sin x) dx &= \int \ln u du \quad \left| \begin{array}{l} u = \sin x \\ du = \cos x dx \end{array} \right. \\ &= u \ln u - \int 1 du \quad \left| \begin{array}{l} F = \ln u \quad G' = 1 \\ F' = \frac{1}{u} \quad G = u \end{array} \right. \\ &= u \ln u - u + C \\ &= \sin x \ln(\sin x) - \sin x + C \end{aligned}$$

$$6. \int \sin \sqrt{x} dx$$

**Work:**

$$\begin{aligned} \int \sin \sqrt{x} dx &= 2 \int u \sin u du \quad \left| \begin{array}{l} u = \sqrt{x} \quad du = \frac{1}{2}x^{-\frac{1}{2}} dx \\ 2x^{\frac{1}{2}} du = dx \Rightarrow 2udu = dx \end{array} \right. \\ &= 2 \left( -u \cos u + \int \cos u du \right) \quad \left| \begin{array}{l} F = u \quad G' = \sin u \\ F' = 1 \quad G = -\cos u \end{array} \right. \\ &= 2(-u \cos u + \sin u) + C \\ &= 2(-\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x}) + C \end{aligned}$$

$$7. \int x^3 e^{x^2} dx \quad \text{This problem is really hard. It's ok to ignore it, I guess.}$$

**Di's Thinking Process:** The first thing that I tried was IBP directly. Since  $e^{x^2}$  is not easy to integrate. I chose  $F = e^{x^2}$ ,  $G = x^3$ . Then  $F' = 2xe^{x^2}$ ,  $G = \frac{x^4}{4}$ . However, look at what happened!

$$\begin{aligned} \int F'G &= \int 2xe^{x^2} \frac{x^4}{4} dx \\ &= \frac{1}{2} \int x^5 e^{x^2} dx \end{aligned}$$

We started with  $x^3$  and now get  $x^5$ , making the problem even harder??!! This shows it's the wrong direction to go! Don't give up! Let's try something else. Well... No other thing could be tried but u-sub. Let's see.

**Work:**

$$\begin{aligned}
 \int x^3 e^{x^2} dx &= \int x^3 e^u \frac{du}{2x} && \left| \begin{array}{l} u = x^2 \\ du = 2x dx \Rightarrow \frac{du}{2x} = dx \end{array} \right. \\
 &= \frac{1}{2} \int x^2 e^u du \\
 &= \frac{1}{2} \int ue^u du && \text{Wow! Magically it worked! We can do IBP now.} \\
 &= \frac{1}{2} \left( ue^u - \int e^u du \right) && \left| \begin{array}{ll} F = u & G' = e^u \\ F' = 1 & G = e^u \end{array} \right. \\
 &= \frac{1}{2} (ue^u - e^u) + C \\
 &= \frac{1}{2} (x^2 e^{x^2} - e^{x^2}) + C && \text{Yeah! We did it!}
 \end{aligned}$$

8.  $I_n = \int x^n \sin(2x) dx$

**Work:**

$$\begin{aligned}
 I_n &= \frac{-\cos(2x)}{2} x^n + \frac{n}{2} \int x^{n-1} \cos(2x) dx && \left| \begin{array}{ll} F = x^n & G' = \sin(2x) \\ F' = nx^{n-1} & G = \frac{-\cos(2x)}{2} \end{array} \right. \\
 &= \frac{-\cos(2x)}{2} x^n + \frac{n}{2} \left( x^{n-1} \frac{\sin(2x)}{2} - \frac{(n-1)}{2} \int x^{n-2} \sin(2x) dx \right) && \left| \begin{array}{ll} F = x^{n-1} & G' = \cos(2x) \\ F' = (n-1)x^{n-2} & G = \frac{\sin(2x)}{2} \end{array} \right. \\
 I_n &= \frac{-\cos(2x)}{2} x^n + \frac{n}{2} x^{n-1} \frac{\sin(2x)}{2} - \frac{n(n-1)}{4} I_{n-2}
 \end{aligned}$$

9.  $I_n = \int x^n e^{2x} dx$

**Work:**

$$\begin{aligned}
 I_n &= \frac{1}{2} x^n e^{2x} - \frac{n}{2} \int x^{n-1} e^{2x} dx && \left| \begin{array}{ll} F = x^n & G' = e^{2x} \\ F' = nx^{n-1} & G = \frac{e^{2x}}{2} \end{array} \right. \\
 I_n &= \frac{1}{2} x^n e^{2x} - \frac{n}{2} I_{n-1}
 \end{aligned}$$

10.  $I_n = \int (1+x)^n \sin x dx$

**Work:**

$$\begin{aligned}
 I_n &= -(1+x)^n \cos x + n \int (1+x)^{n-1} \cos x dx && \left| \begin{array}{ll} F = (1+x)^n & G' = \sin x \\ F' = n(1+x)^{n-1} & G = -\cos x \end{array} \right. \\
 &= -(1+x)^n \cos x + n \int (1+x)^{n-1} \cos x dx && \left| \begin{array}{ll} F = (1+x)^{n-1} & G' = \cos x \\ F' = (n-1)(1+x)^{n-2} & G = \sin x \end{array} \right. \\
 &= -(1+x)^n \cos x + n \left( (1+x)^{n-1} \sin x - (n-1) \int (1+x)^{n-2} \sin x dx \right) \\
 I_n &= -(1+x)^n \cos x + n (1+x)^{n-1} \sin x - n(n-1) I_{n-2}
 \end{aligned}$$

11.  $I_n = \int x^{-n} \sin x dx$

**Work:**

$$\begin{aligned}
 I_n &= -x^{-n} \cos x - n \int x^{-n-1} \cos x & \left| \begin{array}{ll} F = x^{-n} & G' = \sin x \\ F' = -nx^{-n-1} & G = -\cos x \end{array} \right. \\
 &= -x^{-n} \cos x - n \left( x^{-n-1} \sin x + (n+1) \int x^{-n-2} \sin x dx \right) & \left| \begin{array}{ll} F = x^{-n-1} & G' = \cos x \\ F' = (-n-1)x^{-n-2} & G = \sin x \end{array} \right. \\
 I_n &= -x^{-n} \cos x - nx^{-n-1} \sin x + n(n+1) I_{n+2}
 \end{aligned}$$

For reduction formulas, we prefer the integral in the left hand side is of higher order, and the integral in right hand side is lower. That's what we meant by "reduction". But now the order is increasing.

$$I_n + x^{-n} \cos x + nx^{-n-1} \sin x = n(n+1) I_{n+2}$$

$$I_{n+2} = \frac{1}{n(n+1)} (I_n + x^{-n} \cos x + nx^{-n-1} \sin x)$$